

Fluctuations of the Shannon capacity in a Raleigh model of wireless communication

J. Ståring¹, A. Eriksson², and B. Mehlig¹

¹ Theoretical Physics, Physics and Engineering Physics, Göteborg University/Chalmers, Sweden

² Physical Resource Theory, Physics and Engineering Physics, Göteborg University/Chalmers, Sweden

Using the fact that the Shannon capacity C of a Raleigh model of wireless channels is a linear statistic of the channel matrix, we calculate its variance $\text{var}[C]$. We find that the expected value $\langle C \rangle$ of the Shannon capacity is typical in the model considered, that is the coefficient of variation $\sqrt{\text{var}[C]}/\langle C \rangle$ is small.

Copyright line will be provided by the publisher

The efficiency of a wireless channel is determined by its Shannon capacity, $C = \log_2(1 + \rho |H|^2)$ where ρ is the signal-to-noise ratio and H is the transfer characteristic of the channel [1]. The Shannon capacity describes the rate of information transfer (in bits per second, bps). A crucial question in the design of multiple-antenna arrays is: how does the channel capacity increase with the number of channels? Consider an array of n_T transmitters and n_R receivers as shown schematically in Fig. 1. The scattering medium is characterised by a $n_T \times n_R$ channel matrix \mathbf{H} with complex matrix elements H_{kl} determining the amplitude of the l th receiving antenna arriving from transmitter k . In realistic situations, the scattering medium changes as a function of time, and so does the capacity. Typically it fluctuates randomly; it was therefore suggested [2, 1] to calculate an average capacity as an ensemble average over random matrices \mathbf{H} . In [2, 1] an idealised model (called Raleigh model in the following) was considered: H_{kl} were taken to be uncorrelated random variables with zero mean and unit variance. This corresponds to a regular array of antennae, spaced $\lambda/2$ apart (λ is the wave length). In this case, the capacity is given by

$$C(\mathbf{H}) = \log_2 \det \left(\mathbf{1} + \frac{\rho}{n_T} \mathbf{H} \mathbf{H}^\dagger \right) \quad (1)$$

and its average was calculated in [2]. Remarkably, the average capacity was found to scale linearly with the number of transmitters (receivers) for a large number of antennae. This observation has attracted considerable attention, and the average Shannon capacity in related, but more general models (incorporating correlations between the matrix elements H_{kl}) has been studied in great detail [3, 4, 5, 6]. It was found that correlations between the matrix elements H_{kl} reduce the Shannon capacity somewhat, but by increasing the number of antennae, the Shannon capacity can be increased significantly. Empirical studies have indeed shown substantial efficiency gains for such antenna arrays [7].

An important question is however: how typical is the expected value of C ? In other words, how large are its fluctuations? It has been argued that the distribution of the capacity is a sharply peaked function [4]. Recently its distribution was calculated, for the model described in [4], in the limit of large n_T and n_R using the replica technique [8]. In this limit the distribution was found to be Gaussian, with a finite variance.

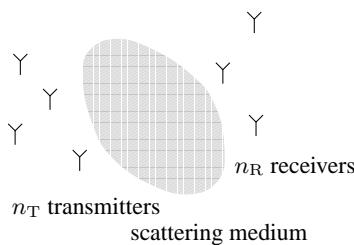


Fig. 1 Wireless array of n_T transmitting and n_R receiving antennae. It is assumed that $n_T \geq n_R$. The scattering medium is characterised by a $n_T \times n_R$ channel matrix \mathbf{H} with complex matrix elements.

Here we calculate the fluctuations of the Shannon capacity for the Raleigh model (1) exactly for arbitrary values of n_T and n_R . We use the fact that the capacity (1) is a *linear statistic* [9] of the channel matrix \mathbf{H} , i.e., it can be expressed in the form $C = \sum_i f(\lambda_i)$ where λ_i are the eigenvalues of the channel matrix. The fluctuations of a linear statistic of a random matrix are determined by the spectral m -point functions. The variance, for example, is given by

$$\text{var}[C] = \int d\lambda \int d\mu K_2(\lambda, \mu) \log_2(1 + \rho\lambda/n_T) \log_2(1 + \rho\mu/n_T) \quad (2)$$

where $K_2(\lambda, \mu) = -\langle \sum_{ij} \delta(\lambda - \lambda_i) \delta(\mu - \lambda_j) \rangle + d(\lambda)d(\mu)$ is the two-point correlation function and $d(\lambda) = \langle \sum_i \delta(\lambda - \lambda_i) \rangle$ is the density of states (the one-point function). Higher moments of C can be expressed in terms of higher spectral correlation functions.

The random matrix ensemble discussed above is in fact the so-called Laguerre ensemble [10]. In this ensemble, the m -point correlation function are known exactly [11]. The two-point correlation function K_2 is usually expressed in terms of the two-level cluster function T_2 , $K_2(\lambda, \mu) = T_2(\lambda, \mu) - d(\lambda)\delta(\lambda - \mu)$, and

$$T_2(\lambda, \mu) = (\lambda\mu)^{a/2} e^{-(\lambda+\mu)/2} \sum_{j=0}^{n_R-1} \frac{j!}{\Gamma(j+a+1)} L_j^a(\lambda) L_j^a(\mu) \quad (3)$$

where $a = n_T - n_R$ and $L_j^a(\lambda)$ are Laguerre polynomials [12]. We have evaluated the variance of the capacity using eqs. (2,3). The results are displayed in Fig. 2a, for the case $n_T = n_R \equiv n$. Shown is the variance of the capacity, $\text{var}[C]$, as a function of the signal-to-noise ratio in the range from zero to 50 dB. For small values of n , the variance is found to depend significantly on n , but for large values of n , it appears to become independent of the number n of antennae. In the limit of large n , we make use of an asymptotic result derived in [13] to obtain, from eqs. (2,3)

$$\text{var}[C] \approx \frac{1}{2\pi^2} \mathcal{P} \int_0^{4n} d\lambda \int_0^{4n} d\mu \left[\frac{\mu(4n-\mu)}{\lambda(4n-\lambda)} \right]^{1/2} \frac{c(\lambda)}{\lambda-\mu} \frac{d}{d\mu} c(\mu) \quad (4)$$

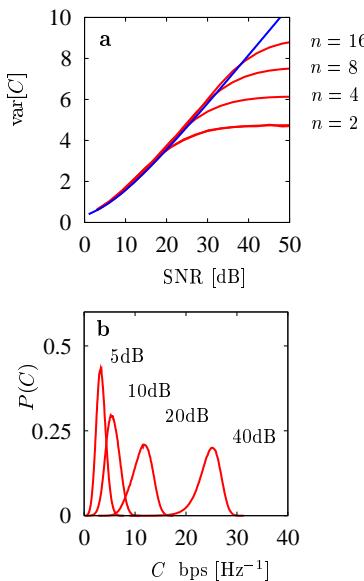


Fig. 2 **a** Variance of the channel capacity as a function of the signal-to-noise ratio (SNR) for $n_R = n_T$, with $n = 2, 4, 8$ and 16 (red, from bottom to top). Also shown is the asymptotic result (5) valid in the limit $n \rightarrow \infty$ (blue). **b** Distribution of the channel capacity for $n_R = n_T = n = 2$ for different signal-to-noise ratios, obtained from diagonalisations of 2×2 matrices.

where $c(\lambda) = \log_2(1 + \rho\lambda/n)$ and \mathcal{P} denotes the principal value. We arrive at

$$\text{var}[C] \approx \frac{1}{\pi(\log 2)^2} \int_0^{\pi/2} d\theta \frac{\log(1+4\rho\sin^2\theta) (1-\sqrt{1+4\rho} + 4\rho\sin^2\theta)}{1+4\rho\sin^2\theta}. \quad (5)$$

This result corresponds to a special case of the asymptotic expression eq. (59) in [8] and is also shown in Fig. 2a. We conclude: for small and moderate signal-to-noise ratios, the asymptotic limit given by eq. (5) is rapidly attained for growing values of n . For large values of ρ , however, the convergence is seen to be much slower. In the limit of $n \rightarrow \infty$, eq. (5) together with the result of [2] gives $\sqrt{\text{var}[C]}/\langle C \rangle \propto n^{-1}$. But even for small values of n we find that the coefficient of variation $\sqrt{\text{var}[C]}/\langle C \rangle$ is small. This implies that the expected value of C is typical even for a small number of antennae; the larger the signal-to-noise ratio is, the more typical is $\langle C \rangle$. Furthermore, since C is a linear statistic, Politzer's argument [14] implies that the distribution of C must be Gaussian in the limit of large n , as noted in [8]. Fig. 2b shows that even for rather low values of n , the distribution is well approximated by a Gaussian for small signal-to-noise ratios ρ . For larger values of ρ , the distribution is seen to develop a tail to the left. This tail, however, is found to rapidly disappear in the limit of large n .

References

- [1] G. J. Foschini and M. J. Gans, *Wireless Personal Communications* **6**, 311 (1998).
- [2] I. E. Telatar, *Bell Lab. Report* 1995.
- [3] Da-Shan Shiu, G. J. Foschini, M. J. Gans, and J. M. Kahn, *IEEE Transactions on Communications* **48**, 502 (2000)
- [4] A. L. Moustakas, H. U. Baranger, L. Balents, A. M. Sengupta, and S. H. Simon, *Science* **287**, 287 (2000)
- [5] S. Loyka, *IEEE Communication Letters*, **5**, 369 (2001)
- [6] S. Loyka and G. Tsoulos, *IEEE Communications Letters* **6**, 19 (2002)
- [7] J. Ling, D. Chizhik, P. Wolniansky, R. Valenzuela, N. Costa, and K. Huber, *IEE Electron. Lett.* **37**, 1041 (2001)
- [8] A. L. Moustakas, S. H. Simon, and A. M. Sengupta, *IEEE Transactions on Information Theory* **49**, 2545 (2003)
- [9] F. J. Dyson and M. L. Mehta, *J. Math. Phys.* **4**, 701 (1963).
- [10] B. V. Bronk, *J. Math. Phys.* **6**, 228 (1965)
- [11] T. Nagao and K. Slevin, *J. Math. Phys.* **34**, 2075 (1993)
- [12] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Academic Press, San Diego (1980)
- [13] C. W. J. Beenakker, *Nucl. Phys. B* **422**, 515 (1994)
- [14] H. D. Politzer, *Phys. Rev. B* **40**, 11917 (1989)